D5-17266-1 Addendum to Volume 1 of 3

# SSME STRUCTURAL COMPUTER PROGRAM DEVELOPMENT

## BOPACE THEORETICAL MANUAL - ADDENDUM
CONTRACT NAS8-30615

APRIL 15, 1975

(NASA-CR-143893) SSME STRUCTURAL COMPUTER PROGRAM DEVELOPMENT: BOPACE THEORETICAL MANUAL, ADDENDUM (Boeing Aerospace Co., Seattle, Wash.) 14 p HC $3.25 CSCL 09B
N75-27777
Unclas
G3/60 29185

Prepared by

BOEING AEROSPACE COMPANY
RESEARCH AND ENGINEERING DIVISION
SEATTLE, WASHINGTON 98124



R. G. VOS - TECHNICAL LEADER

J. W. STRAAYER - PROGRAM MANAGER

Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
GEORGE C. MARSHALL SPACE FLIGHT CENTER
MARSHALL SPACE FLIGHT CENTER, ALABAMA 35812

# BOPACE THEORETICAL MANUAL - ADDENDUM

## 2.6 IMPROVED ALGORITHM FOR INELASTIC CALCULATIONS

Summary of Basic Concepts - The iterative residual-force procedure is often employed with an incremental solution for inelastic (plasticity and creep) problems, in order to avoid accumulated error. Each iteration in the residual-force procedure involves the following two stages.

1) Equilibrium and Compatibility: Given the current residuals (unbalanced forces or stresses), the equilibrium and compatibility equations are applied in order to predict an improved configuration (of displacements and strains).

2) eparation of Strains: Given the current strains, some algorithm based on the inelastic material theory is applied in order to separate the strains into their elastic, plastic and creep portions, and thus provide the resulting stresses.

When this procedure has converged to the correct result, the following conditions will be met.

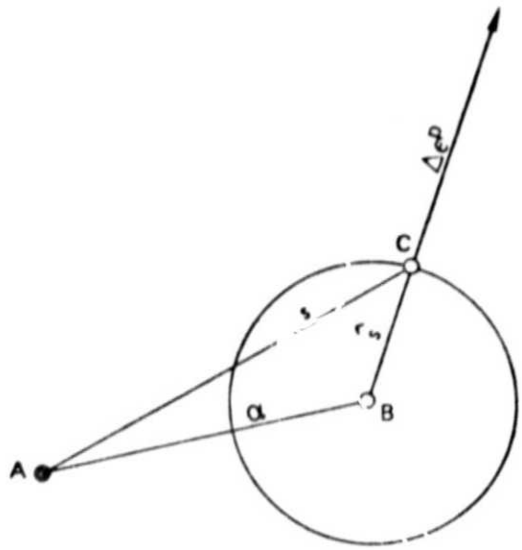
1) Forces in equilibrium
2) Displacements compatible
3) Plastic strain increment satisfies normality rule
4) Size of yield surface consistent with deformation history
5) Translation of yield surface consistent with deformation history

The overall BOPACE solution technique based on the residual-force procedure is summarized in Section 4. The purpose of the present section is to discuss the details of a new algorithm which has been developed and incorporated into BOPACE, for improving the convergence and accuracy of the inelastic stress-strain calculations. This algorithm defines the implementation of stage 2 (separation of strains) in the residual-force iterative procedure.

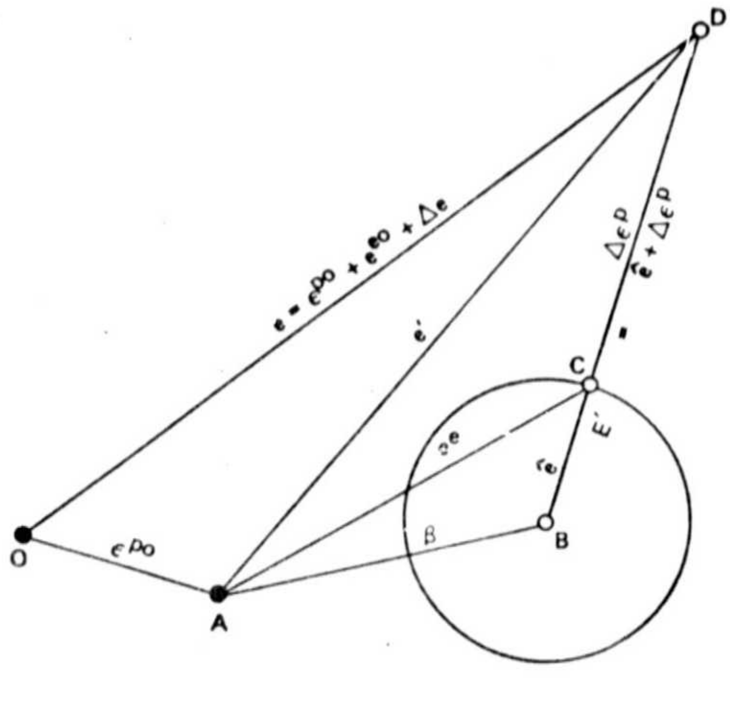
Background - The theory already presented in Sections 2.1 through 2.5 may be employed for both stages of the iterative procedure, and in fact equations of the type 2.5-5 were used for all stress-strain calculations in the initial version of BOPACE. Convergence difficulties resulted from the use of this approach in stage 2, however, when the incremental inelastic strains were large relative to the cumulative elastic strains. These difficulties were substantially eliminated by properly controlling the direction defined for the incremental inelastic strains. (The reason for the difficulties and the method of control were presented in Reference A-1). Another quite different approach is based on a "strain-space" concept, and was presented by Barsoum in Reference A-2 with the claim of a significant improvement in efficiency. That approach has therefore also been developed and evaluated for BOPACE usage. Because the method as presented in Reference A-2 assumes kinematic hardening only, it was extended to incorporate the combined isotropic and kinematic hardening provided by BOPACE. In addition, some further techniques for accelerating convergence were identified and incorporated into the method. The resulting modified version of BOPACE has shown promising results. For example, a small test problem involving 15 load increments and solved using BOPACE solution option 5, required 25% fewer iterations and 50% fewer stiffness matrix updates than when run on the previous version. Although further comparisons are needed to prove the effectiveness of the new algorithm, it is expected to supersede the previous BOPACE capability.

Basic Definitions and Comparison of Algorithms - The new inelastic algorithm involves calculations in the "deviatoric strain space", rather than the more conventional "deviatoric stress space" used in previous BOPACE programs. For the sake of clarity, the previously used stress-space algorithm will again be summarized here, and the elastic-plastic quantities used in the new strain-space algorithm will be defined and compared with previous quantities.

Note: ● = Point which is fixed during increment
O = Point which moves during increment




a) QUANTITIES IN DEVIATORIC STRESS SPACE




b) QUANTITIES IN DEVIATORIC STRAIN SPACE

*Figure 2.6-1. Graphical Representation of Elastic–Plastic Quantities*

As described in Section 2.3, the definition of a plasticity theory requires assumptions for three basic constituents: a yield surface, a flow rule, and a hardening assumption. BOPACE development is based on the Mises yield surface, and this surface is represented by a hypercircle in 9-dimensional deviatoric stress space, as shown in Figure 2.6-1a. The surface is defined by the equation

$$F = \hat{s}_i \hat{s}_i - \hat{s}^o_i \hat{s}^o_i = 0 \tag{2.6-1}$$

where s is the deviatoric stress, $\hat{s} = s - \alpha$ is the relative deviatoric stress and defines the isotropic hardening, $\alpha$ is the surface translation and defines the kinematic hardening, while $\hat{s}^o$ is a reference value of $\hat{s}$ and must be known as a function of plastic deformation (e.g. from a uniaxial test). Point A in Figure 2.6-1a is the origin of the deviatoric stress space, point B is the current center of the yield surface, and point C represents the current state of deviatoric stress. A stress point on the surface corresponds to a plastic state. According to the Prandtl-Reuss flow rule, the direction of the incremental plastic strain, $\Delta\varepsilon^P$, is normal to the yield surface at the current deviatoric stress state, s. A solid circle ( ● ) in Figure 2.6-1 denotes a point which remains fixed throughout the increment, while an open circle ( O ) denotes a point which moves during the increment. In order to achieve greater accuracy and allow larger load increments, BOPACE evaluates moving points such as B and C at the midpoint of the plastic increment. Additional details of the BOPACE stress-space algorithm are discussed in Section 2.3.

For the new strain-space algorithm, the three basic constituents of the plasticity theory remain unchanged, and direct use is made of the stress-space theory and nomenclature. However, we now work with a yield surface and associated quantities in strain-space. Thus we compute the deviatoric elastic strain, $e^e$, in terms of the deviatoric stress, s, by

$$e_i^e = s_i/G \tag{2.6-2}$$

where $G = E/(1+\nu)$ is a tensorial shear modulus. Similarly we define a "strain center", $\beta$, in terms of the stress center, $\alpha$, by

$$\beta_i = \alpha_i/G \tag{2.6-3}$$

Then the relative deviatoric strain, $\hat{e}$, is defined by

$$\hat{e}_i = e_i^e - \beta_i = (s_i - \alpha_i)/G = \hat{s}_i/G \tag{2.6-4}$$

The geometrical interpretation of the new algorithm involving these quantities is provided by a sketch in 9-dimensional deviatoric strain-space, shown in Figure 2.6-1b. There point 0 is the origin, defining the initial undeformed (zero strain) state. Subsequent deformation is caused by a series of load increments, resulting in elastic and plastic strains. A superscript 0 is used to denote the value of a quantity at the beginning of the load increment. Thus, point A defines the cumulative plastic strain, $\varepsilon^{p0}$, which exists at the beginning of the current increment.

(Because of the plastic incompressibility assumption, the plastic strains themselves are deviatoric strains). All other points in Figure 2.6-1b refer to locations at some time during the current increment. In particular, we will be mainly concerned with the location of these points at a defined reference time. This reference time may be taken at the end of the increment, following the approach of Barsoum [14], or greater accuracy may be obtained at the expense of some additional variable storage by taking the reference time at the midpoint of the plastic increment, as is done in the new BOPACE algorithm. Point D defines the total cumulative deviatoric strain, e, during the increment. The circle is associated with the Mises yield surface, but is a hyper-circle in the deviatoric strain space. A strain point within the surface corresponds to an elastic state, while a strain point outside the surface corresponds to a plastic state. The size of this circle is defined by its radius $\hat{e}_i (\hat{e}_i = \hat{s}_i/G)$, whereas the Mises stress-space surface has radius $\hat{s}$. The center of the circle is at point B ($B_i = \epsilon_i^{p0} + \beta_i = \epsilon_i^{p0} + \alpha_i/G$), whereas the center of the Mises stress-space surface has components $\alpha_i$. During plastic deformation, the strain-space surface may undergo both expansion (due to isotropic hardening), and translation (due to kinematic hardening). The cumulative deviatoric elastic strain, $e^e$, is defined by the vector AC ($e_i^e = s_i/G$). From these comparisons it should be apparent that the basic quantities in Figures 2.6-1a and b, respectively, can be made to coincide, if points A are superimposed and all dimensions in 2.6-1b are divided by the factor G. The incremental plastic strain, $\Delta\epsilon^p$, is defined by the vector CD. It is normal to the circle because of the Prandtl-Reuss flow rule, and is therefore colinear with the radius $\hat{e}$ to point C. The vector BD = $\hat{e} + \Delta\varepsilon^p$ is denoted by E'. The symbols e', $e^e$, $\Delta\varepsilon^p$ and E' are consistent with their usage in Reference A-2.

Computation Procedure - We now define the new strain-space algorithm for implementing stage 2 of the residual-force iterative procedure. The problem which must be solved can be stated in terms of the various strain vectors. At the beginning of the increment, we have known values for $\varepsilon^{p0}$ (which remains constant during the increment), and for $\beta$, $e^e$, and $\hat{e}$. These have been determined such that they are all consistent, i.e., such that the appropriate vectors meet at single points A, B and C. The current estimate for the value of e' at the reference time is also known from stage 1 of the iterative procedure. We must then determine values for $\beta$, $e^e$, $\hat{e}$ and $\Delta\varepsilon^p$ at the reference time, consistent with the convergence requirements. Stated somewhat differently, we are given the locations of points A and D at the reference time, and the locations of points B and C at the beginning of the increment. We must then compute the locations of B and C at the reference time, consistent with the convergence requirements.

The basic steps of the stage 2 algorithm are summarized by the following.

1) Given values at beginning of increment for:

$\alpha^0$ = stress center

$\hat{s}^0$ = relative deviatoric stress

$e^{e0}$ = elastic strains

2) Given $\Delta\varepsilon$ = total (elastic + plastic) strain increment from stage 1.

3) Compute values at reference time, based on estimated incremental deformation, for:

$\Delta\alpha$ = kinematic hardening increment

$\Delta\hat{s}$ = isotropic hardening increment

4) Compute: $\varepsilon_i = \varepsilon_i^{e0} + \Delta\varepsilon_i$ = initial elastic strain + total strain increment

$e'_i$ = corresponding deviatoric value

5) Compute: $\beta_i = (\alpha_i^0 + \Delta\alpha_i)/G$

$\hat{e}_i = (\hat{s}_i^0 + \Delta\hat{s}_i)/G$

6) Compute $E'_i = e'_i - \beta_i$

7) Compute $\lambda = (|E'| - |\hat{e}|)/|E'|$ = plastic proportionality constant

8) Compute $\Delta\varepsilon_i^p = \lambda E'_i$ = incremental plastic strain at reference time. Adjust $\Delta\varepsilon_i^p \leftarrow \Delta\varepsilon_i^p$ times (ratio of total to reference time increment), to obtain total plastic strain increment. Compute $\Delta\varepsilon_i^e = \Delta\varepsilon_i - \Delta\varepsilon_i^p$ = incremental elastic strain.

9) Compute end of increment values for:

$\varepsilon_i^e = \varepsilon_i^{e0} + \Delta\varepsilon_i^e$ = cumulative elastic strain

$\sigma_i = D_{ij}\ \varepsilon_j^e$ = cumulative stress

10) Use $\sigma$ to compute residual forces and error norm, and return to stage 1 if convergence has not been achieved.

The strain-space algorithm presented above corresponds to that given by Barsoum [A-2] except that here a combined isotropic and kinematic hardening is provided and a reference (midpoint) time calculation of the incremental variables is used to improve accuracy. As noted by Barsoum, greater consistency and better convergence are obtained by utilizing an algorithm in strain space rather than in stress space. This is because the stress-space calculation fixes the $\Delta\epsilon^p$ vector along the direction of the current $\hat{s}$ vector, rather than simultaneously fixing the directions of $\hat{s}$ and $\Delta\epsilon^p$ consistent with the given total strain increment $\Delta\epsilon$. The stress-space iteration can cause large oscillations in the location of point C, resulting in divergence if $\Delta\epsilon^p$ is large relative to the cumulative elastic strain.

Although a strain-space algorithm eliminates most of the inconsistencies and tendencies toward divergence, it should be noted that an inconsistency still exists in the plastic hardening quantities. This is because $\Delta\alpha$ and $\Delta\hat{s}$ are based on the estimated increment of plastic deformation, which will not in general be consistent with the actual deformation. Thus if another iteration were performed using the same value for the total strain increment $\Delta\epsilon$, different results would be obtained due to change in $\beta$ and $\hat{e}$.

This difficulty is eliminated in the present approach by properly modifying the calculation of $\lambda$ in step 7. For this calculation, we use the parameters c and r associated with kinematic and isotropic hardening, respectively, in the expressions

$$
\begin{aligned}
\Delta\alpha_i &= \frac{2}{3}\, c\, \Delta\varepsilon_i^p \\
\Delta\hat{s}_i &= \frac{2}{3}\, r\, \Delta\varepsilon_i^p
\end{aligned}
\qquad (2.6\text{-}5)
$$

We then see that

$$
\begin{aligned}
E_i' &= e_i' - \beta_i = e_i' - (\beta_i^0 + \Delta\beta_i) \\
&= e_i' - \beta_i^0 - \Delta\alpha_i/G = e_i' - \beta_i^0 - \frac{2}{3}\, c\, \Delta\varepsilon_i^p/G
\end{aligned}
\qquad (2.6\text{-}6)
$$

Replacing $\Delta\varepsilon_i^p$ in this equation by $\lambda E_i'$, we may solve for $E_i'$:

$$
E_i' = (e_i' - \beta_i^0)/(1 + \frac{2}{3}\lambda\, c\, /G) \qquad (2.6\text{-}7a)
$$

In a similar manner we may obtain

$$\hat{e}_i = \hat{e}_i^0 + \frac{2}{3} \lambda\, r\, E_i'/G \qquad (2.6\text{-}7b)$$

The plastic proportionality constant, as already defined, is

$$\lambda = (|E'| - |\hat{e}|)/|E'| \qquad (2.6\text{-}7c)$$

It is apparent from Equations 2.6-7 that the expression for $\lambda$ is non-linearly dependent upon $\lambda$ itself, and this is the reason why a consistent $\lambda$ is not solved for directly. An accurate value for $\lambda$, however, can easily be obtained by a "linear intersection method." In this method we take the approximate value of $\lambda$ from step 7, and substitute into the Equations 2.6-7 to obtain a new computed value $\lambda_{c0}$. We then assume a value of $\lambda + \Delta\lambda$, where $\Delta\lambda$ is a small change (perhaps $.01\lambda$), and again substitute into Equations 2.6-7 to compute another value $\lambda_{c1}$. The two pairs of assumed and computed $\lambda$ values are plotted in Figure 2.6-2. The correct value for $\lambda$ lies on the 45-degree line (since there the assumed and computed values would be equal), at the intersection of this line with the line connecting the two plotted points. This corrected value of $\lambda$ is obtained by the following adjustment of $\lambda$ from step 7.

$$\lambda \leftarrow \lambda + \Delta\lambda(\lambda - \lambda_{c0})/(\lambda_{c1} - \lambda_{c0} - \Delta\lambda) \qquad (2.6\text{-}8)$$

The incorporation of this adjustment into the strain-space algorithm provides consistent values for all quantities in stage 2 of the iterative process, and results in improved convergence.

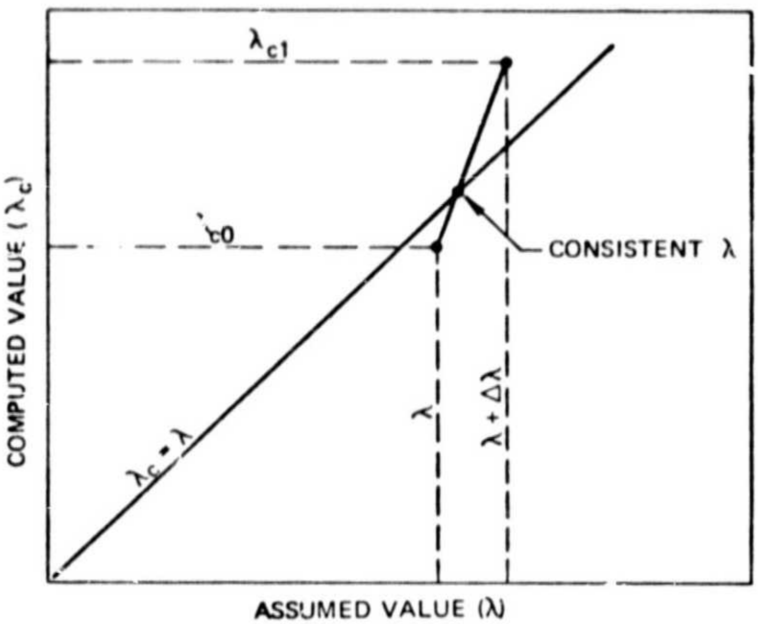



*Figure 2.6-2. Linear-Intersection Calculation for λ*

Extensions and Refinements to the Basic Algorithm - The strain-space algorithm as presented here is employed in BOPACE for plastic analysis. In addition, the BOPACE algorithm treats creep strains in a manner similar to that for the plastic strains. For cases where the material is elastic at the beginning of an increment and then reaches the plastic yield point at some intermediate time during the increment, greater accuracy is obtained by dividing the calculations into two parts. In such cases the initial creep is taken in the direction of the initial deviatoric stress, and creep which occurs after the yield point is taken in the same direction as the plastic strain increment. Other refinements, such as temperature dependent elastic-plastic-creep and generalized load reversal, are treated as discussed in Section 2.

REFERENCES

A-1. R. G. Vos, "Note on the Residual Force Method," Comment on a paper by G. C. Nayak and O. C. Zienkiewicz, International Journal for Numerical Methods in Engineering, Vol. 6, No. 3, 1973.

A-2. R. S. Barsoum, "A Convergent Method for Cyclic Plasticity Analysis with Application to Nuclear Components," International Journal for Numerical Methods in Engineering, Vol. 6, No. 2, 1973.